Schrödinger–Wheeler–DeWitt Equation in Chaplygin Gas FRW Cosmological Model

P. Pedram · S. Jalalzadeh · S.S. Gousheh

Abstract We present a Chaplygin gas Friedmann–Robertson–Walker quantum cosmological model. In this work the Schutz's variational formalism is applied with positive, negative, and zero constant spatial curvature. In this approach the notion of time can be recovered. These give rise to Schrödinger–Wheeler–DeWitt equation for the scale factor. We use the eigenfunctions in order to construct wave packets for each case. We study the time dependent behavior of the expectation value of the scale factor, using the many-worlds interpretations of quantum mechanics.

1 Introduction

In recent years supernova Ia (SNIa) observations show that the expansion of the universe is accelerating [1–3] contrary to Friedmann–Robertson–Walker (FRW) cosmological models, with non-relativistic matter and radiation. Also cosmic microwave background radiation (CMBR) data [4–6] is suggesting that the expansion of our universe seems to be in an accelerated state. This is referred to "dark energy" effect [7]. Cosmological constant, Λ, as usual vacuum energy can be responsible for this evolution by providing a negative pressure [8, 9]. Unfortunately, the observed value of Λ is 120 orders of magnitude smaller than the one computed from field theory methods [8, 9]. Quintessence is an alternative to consider a dynamical vacuum energy [10–14], involving one or two scalar fields, some with potentials justified from supergravity theories [15]. However, the fine-tuning problem of these models which arise from cosmic coincidence issue has no satisfactory solution.

The Chaplygin gas model is an interesting proposal [16], describing a transition from a universe filled with dust-like matter to an accelerating expanding stage. This model was later generalized in Refs. [17–19]. The generalized Chaplygin gas model is described by a perfect fluid obeying an exotic equation of state [19]

$$p = -\frac{A}{\rho^\alpha}, \tag{1}$$

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where A is a positive constant and $0 < \alpha \leq 1$. The standard Chaplygin gas [16] corresponds to $\alpha = 1$. Some publications [20–43] and reviews [40, 41] which studied the Chaplygin gas cosmological models have already appeared in the literature.

Recently, quantum mechanical description of a FRW model with a generalized Chaplygin gas has been discussed in Ref. [44] in order to retrieve explicit mathematical expressions for the different quantum mechanical states and determine the transition probabilities towards an accelerated stage. In this paper we investigate the existence of singularities at quantum level in Chaplygin gas cosmological models. In the quantum cosmology the Wheeler–DeWitt (WD) equation in minisuperspace which determines the wave function of the Universe, can be constructed using ADM decomposition of the geometry [45] in the Hamiltonian formalism of general relativity.

The presence of matter in quantum cosmology needs further consideration and can be described by fundamental fields, as done in Ref. [46]. Using WKB approximation one can predict the behavior of the quantum universe which leads to determination of the trajectories in phase space. However, even in the minisuperspace, general exact solutions are hard to find, the Hilbert space structure is obscure and it is a subtle matter to recover the notion of a semiclassical time [46, 47].

In the present work, we describe matter as a Chaplygin gas. This description is essentially semiclassical from the start, but it has the advantage of furnishing a variable, connected with the matter degrees of freedom, which can naturally be identified with time, leading to a well-defined Hilbert space structure. It is very convenient to construct a quantum Chaplygin gas model. Schutz's formalism [48, 49] gives dynamics to the fluid degrees of freedom in interaction with the gravitational field. Using proper canonical transformations, at least one conjugate momentum operator associated with matter appears linearly in the action integral. Therefore, a Schrödinger-like equation can be obtained with the matter variable playing the role of time.

Here, we use the formalism of quantum cosmology in order to quantize three Friedmann–Robertson–Walker Chaplygin gas models. In Sect. 2 the quantum cosmological model with a Chaplygin gas as the matter content is constructed in Schutz's formalism, and the Schrödinger–Wheeler–DeWitt (SWD) equation in minisuperspace is written down to quantize the model. The wave-function depends on the scale factor a and on the canonical variable associated to the fluid, which in the Schutz variational formalism plays the role of time T. We separate the wave-function into two parts, one depending solely on the scale factor and the other depending only on the time. The solution in the time sector of the SWD equation is trivial, leading to imaginary exponentials of the type e^{-iEt}, where E is the system energy and $t = -T$. In Sect. 3 we construct wave packets from the eigenfunctions and compute the time-dependent expectation values of the scale factors. In Sect. 4, we present our conclusions.

2 Model

We need the Hamiltonian for a Chaplygin gas model in the formalism developed by Schutz. The starting point is the action for gravity plus Chaplygin gas, which in this formalism is written as

$$S = \int_M d^4x \sqrt{-g}\, R + 2 \int_{\partial M} d^3x \sqrt{h}\, h_{ab}\, K^{ab} + \int_M d^4x \sqrt{-g}\, p, \tag{2}$$

where K^{ab} is the extrinsic curvature and h_{ab} is the induced metric over the three-dimensional spatial hypersurface, which is the boundary ∂M of the four dimensional manifold M. Units are chosen such that the factor $16\pi G$ becomes equal to one. The first two terms were first obtained in [45]; the last term of (2) represents the matter contribution to the total action and p is the pressure. In Schutz's formalism [48, 49] the fluid's four-velocity is expressed in terms of five potentials Φ, ζ, β, θ and S

$$u_\nu = \frac{1}{\mu}(\Phi_{,\nu} + \zeta\beta_{,\nu} + \theta S_{,\nu}) \tag{3}$$

where μ is the specific enthalpy. The variable S is the specific entropy, while the potentials ζ and β are connected with rotation and are absent in models of the Friedmann–Robertson–Walker (FRW) type. The variables Φ and θ have no clear physical meaning. The four-velocity is subject to the normalization condition

$$u^\nu u_\nu = -1. \tag{4}$$

The FRW metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)g_{ij}dx^i dx^j, \tag{5}$$

is now inserted in the action (2). In this expression, $N(t)$ is the lapse function and g_{ij} is the metric on the constant-curvature spatial section.

Following the thermodynamic description of Ref. [50], the basic thermodynamic relations take the form

$$\rho = \rho_0[1 + \Pi], \qquad h = 1 + \Pi + p/\rho_0 \tag{6}$$

$$\tau dS = d\Pi + p\,d(1/\rho_0) = \frac{(1+\Pi)^{-\alpha}}{1+\alpha}d\left[(1+\Pi)^{1+\alpha} + \frac{A}{\rho_0^{1+\alpha}}\right]. \tag{7}$$

It then follows that to within a factor

$$\tau = \frac{(1+\Pi)^{-\alpha}}{1+\alpha} \tag{8}$$

$$S = (1+\Pi)^{1+\alpha} + \frac{A}{\rho_0^{1+\alpha}}. \tag{9}$$

Therefore, the equation of state takes the form

$$p = -A\left[\frac{1}{A}\left(1 - \frac{h^{\frac{\alpha+1}{\alpha}}}{S^{1/\alpha}}\right)\right]^{\frac{\alpha+1}{\alpha}}. \tag{10}$$

The energy density and particle number density are, respectively,

$$\rho = \left[\frac{1}{A}\left(1 - \frac{h^{\frac{\alpha+1}{\alpha}}}{S^{1/\alpha}}\right)\right]^{\frac{-1}{1+\alpha}} \tag{11}$$

$$\rho_0 = \frac{\rho + p}{h} \tag{12}$$

where $h = (\dot{\Phi} + \theta \dot{S})/N$. After dropping the surface terms, the final reduced action takes the form

$$S = \int dt \left\{ -6\frac{\dot{a}^2 a}{N} + 6kNa - Na^3 A \left[\frac{1}{A} \left(1 - \frac{(\dot{\Phi} + \theta \dot{S})^{\frac{\alpha+1}{\alpha}}}{N^{\frac{\alpha+1}{\alpha}} S^{1/\alpha}} \right) \right]^{\frac{\alpha+1}{\alpha}} \right\}. \tag{13}$$

The reduced action may be further simplified using canonical methods [50], resulting in the super-Hamiltonian

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6ka + \left(S p_\Phi^{1+\alpha} + A a^{3(1+\alpha)} \right)^{\frac{1}{1+\alpha}} \tag{14}$$

where $p_a = -12\dot{a}\,a/N$ and $p_\Phi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}}$. However, an analytical quantum mechanical treatment of this FRW minisuperspace with the above Hamiltonian does not seem feasible. Therefore, it requires the following approximation [44],

$$(S p_\Phi^{1+\alpha} + A a^{3(1+\alpha)})^{\frac{1}{1+\alpha}}$$
$$\approx S^{\frac{1}{1+\alpha}} p_\Phi \left[1 + \frac{1}{1+\alpha} \frac{A a^{3(\alpha+1)}}{S p_\Phi^{1+\alpha}} + \frac{1}{2} \frac{1}{1+\alpha} \left(\frac{1}{1+\alpha} - 1 \right) \frac{A^2}{S^2 p_\Phi^{2(1+\alpha)}} a^{6(\alpha+1)} + \cdots \right]. \tag{15}$$

Hence, up to the leading order, the super-Hamiltonian takes the form

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6ka + S^{\frac{1}{1+\alpha}} p_\Phi. \tag{16}$$

The following additional canonical transformations,

$$T = -(1+\alpha) p_\Phi^{-1} S^{\frac{\alpha}{1+\alpha}} p_S, \qquad p_T = S^{\frac{1}{1+\alpha}} p_\Phi, \tag{17}$$

simplifies the super-Hamiltonian to,

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6ka + p_T, \tag{18}$$

where the momentum p_T is the only remaining canonical variable associated with matter. It appears linearly in the super-Hamiltonian. The parameter k defines the curvature of the spatial section, taking the values $0, 1, -1$ for a flat, positive-curvature or negative-curvature Universe, respectively.

The classical dynamics is governed by the Hamilton equations, derived from (18) and Poisson brackets, namely

$$\begin{cases} \dot{a} = \{a, N\mathcal{H}\} = -\dfrac{Np_a}{12a}, \\[2mm] \dot{p}_a = \{p_a, N\mathcal{H}\} = -\dfrac{N}{24a^2} p_a^2 + 6Nk, \\[2mm] \dot{T} = \{T, N\mathcal{H}\} = N, \\[2mm] \dot{p}_T = \{p_T, N\mathcal{H}\} = 0. \end{cases} \tag{19}$$

We also have the constraint equation $\mathcal{H} = 0$. Choosing the gauge $N = 1$, we have the following solutions for the system

$$\ddot{a} = -\frac{\dot{a}^2}{2a} - \frac{k}{2a}, \tag{20}$$

$$0 = -6a\dot{a}^2 - 6ka + p_T. \tag{21}$$

Imposing the standard quantization conditions on the canonical momenta and demanding that the super-Hamiltonian operator annihilate the wave function, we are led to the following SWD equation in minisuperspace ($\hbar = 1$)

$$\frac{\partial^2 \Psi}{\partial a^2} - 144ka^2\Psi + i24a\frac{\partial \Psi}{\partial t} = 0. \tag{22}$$

In this equation, $t = -T$ corresponds to the time coordinate. As discussed in [51, 52], in order for the Hamiltonian operator $\hat{H}$ to be self-adjoint the inner product of any two wave functions Φ and Ψ must take the form

$$(\Phi, \Psi) = \int_0^\infty a\Phi^*\Psi \, da. \tag{23}$$

Moreover, the wave functions should satisfy the restrictive boundary conditions

$$\Psi(0, t) = 0 \quad \text{or} \quad \left.\frac{\partial \Psi(a, t)}{\partial a}\right|_{a=0} = 0. \tag{24}$$

The SWD equation (22) can be solved by separation of variables as follows

$$\psi(a, t) = e^{-iEt}\psi(a) \tag{25}$$

where the a dependent part of the wave function ($\psi(a)$) satisfies

$$-\psi''(a) + 144ka^2\psi(a) = 24Ea\,\psi(a), \tag{26}$$

and the prime means derivative with respect to a.

3 Results

For $k = 0$ the time-independent Wheeler–DeWitt equation (26) reduces to

$$\psi'' + 24Ea\psi = 0. \tag{27}$$

The Bessel functions are solutions of the above equation. Therefore, the time dependent solutions are as follows

$$\Psi_E = e^{-iEt}\sqrt{a}\left[c_1 J_{\frac{1}{3}}\left(\frac{\sqrt{96E}}{3}a^{\frac{3}{2}}\right) + c_2 Y_{\frac{1}{3}}\left(\frac{\sqrt{96E}}{3}a^{\frac{3}{2}}\right)\right]. \tag{28}$$

Now, the wave packets can be constructed, by superposing these eigenfunctions with the following structure

$$\Psi(a, t) = \int_0^\infty A(E)\Psi_E(a, t)dE. \tag{29}$$

Fig. 1 The behavior of the expected value for the scale factor $\langle a \rangle(t)$ (*solid line*) and the classical scale factor $a(t)$ (*dashed line*)
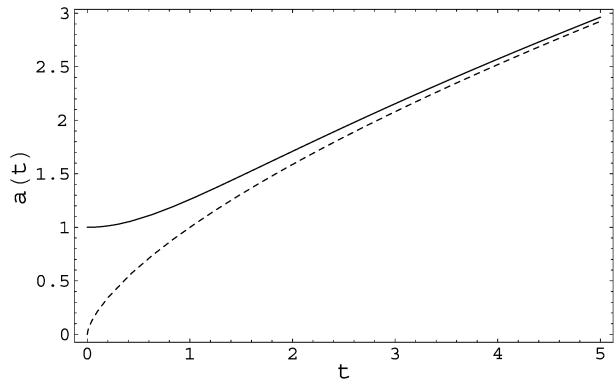
We choose $c_2 = 0$, for satisfying the first boundary condition (24). By choosing $A(E)$ as a quasi-Gaussian weight factor and defining $r = \frac{\sqrt{96E}}{3}$, analytical expressions for the wavepacket can be found

$$\Psi(a,t) = \sqrt{a} \int_0^\infty r^{\nu+1} e^{-\gamma r^2 + i\frac{3}{32}r^2 t} J_\nu(ra^{\frac{3}{2}}) dr, \tag{30}$$

where $\nu = \frac{1}{3}$ and γ is an arbitrary positive constant. The above integral is known [53], and the wave packet takes the form

$$\Psi(a,t) = a \frac{e^{-\frac{a^3}{4B}}}{(-2B)^{\frac{4}{3}}}, \tag{31}$$

where $B = \gamma - i\frac{3}{32}t$. Following the many worlds interpretation of quantum mechanics [54], we may write the expected value for the scale factor a as

$$\langle a \rangle(t) = \frac{\int_0^\infty a\Psi(a,t)^* a\Psi(a,t) da}{\int_0^\infty a\Psi(a,t)^* \Psi(a,t) da}. \tag{32}$$

which yields

$$\langle a \rangle(t) \propto \left[\frac{9}{(32)^2 \gamma^2} t^2 + 1 \right]^{\frac{1}{3}}. \tag{33}$$

These solutions represent a bouncing Universe, with no singularity, which goes asymptotically to the corresponding flat classical models for late times (Fig. 1)

$$a(t) \propto t^{2/3}. \tag{34}$$

In the case $k = 1$ the time-independent Wheeler–DeWitt equation (26) reduces to

$$-\psi''(a) + (-24Ea + 144a^2)\psi(a) = 0. \tag{35}$$

Defining new variable $x = 12a - E$ we find

$$-\frac{d^2\psi}{dx^2} + \left[-\frac{E^2}{144} + \frac{x^2}{144} \right] \psi(a) = 0. \tag{36}$$

Equation (36) is formally identical to the time-independent Schrödinger equation for a harmonic oscillator with unit mass and energy λ

$$-\frac{d^2\psi}{dx^2} + [-2\lambda + w^2 x^2]\psi(x) = 0, \tag{37}$$

where $2\lambda = E^2/144$ and $w = 1/12$. As much as the allowed values of λ are $n + 1/2$, the possible values of E are

$$E_n = \sqrt{12(2n+1)}, \quad n = 0, 1, 2, \ldots. \tag{38}$$

Thus, the stationary solutions are

$$\Psi_n(a, t) = e^{-iE_n t}\varphi_n(12a - E_n), \tag{39}$$

where

$$\varphi_n(x) = H_n\left(\frac{x}{\sqrt{12}}\right)e^{-x^2/24}, \tag{40}$$

with H_n the n-th Hermite polynomial. The wave functions (39) are similar to the stationary quantum wormholes as defined in [55]. However, neither of the boundary conditions (24) can be satisfied by the these wave functions.

In the $k = -1$ case the equation (26) reduces to

$$\psi''(a) + (24Ea + 144a^2)\psi(a) = 0, \tag{41}$$

where the solutions are

$$\Psi(a, t) = e^{-iEt}(12a + E)^{-1/2}$$
$$\times \left\{ C_1 M_{\frac{iE^2}{48}, \frac{1}{4}}\left(\frac{i(12a + E)^2}{12}\right) + C_2 W_{\frac{iE^2}{48}, \frac{1}{4}}\left(\frac{i(12a + E)^2}{12}\right) \right\} \tag{42}$$

where $M_{\kappa,\lambda}$ and $W_{\kappa,\lambda}$ are Whittaker functions. The Whittaker functions do not automatically vanish at $a = 0$. Therefore, in order to satisfy $\Psi(0, t) = 0$ it is necessary to take both $C_1 \neq 0$ and $C_2 \neq 0$, the same is applied to the second of the boundary conditions (24).

4 Conclusions

In this work we have investigated closed, flat, and open minisuperspace FRW quantum cosmological models ($k = 1, 0, -1$) with Chaplygin gas Universes. The use of Schutz's formalism for Chaplygin gas allowed us to obtain a SWD equation in which the only remaining matter degree of freedom played the role of time. We have obtained eigenfunctions and therefore acceptable wave packets were constructed by appropriate linear combination of these eigenfunctions. The time evolution of the expectation value of the scale factor has been determined in the spirit of the many-worlds interpretation of quantum cosmology.

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